

An L^2 -isolation theorem for Yang-Mills fields on Kähler surfaces

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Abstract

We prove an L^2 energy gap result for Yang-Mills connections on principal G -bundles over compact Kähler surfaces with positive scalar curvature. We prove related results for compact simply-connected Calabi-Yau 2-folds.

Keywords. Yang-Mills connection; anti-self-dual connection; Kähler surface

1 Introduction

A connection on a principal bundle is called Yang-Mills connection when it gives a critical point of the Yang-Mills functional, that is, it satisfies the Yang-Mills equation

$$d_A^* F_A = 0.$$

From the Bianchi identity $d_A F_A = 0$, a Yang-Mills connection is nothing but a connection whose curvature is harmonic with respect to the covariant exterior derivative d_A . In this article the principal bundle P is always smooth.

Over a 4-dimensional Riemannian manifold, F_A is decomposed into its self-dual and anti-self-dual components,

$$F_A = F_A^+ + F_A^-$$

where F_A^\pm denotes the projection onto the ± 1 eigenspace of the Hodge star operator. A connection is called self-dual (respectively anti-self-dual) if $F_A = F_A^+$ (respectively $F_A = F_A^-$). A connection is called an instanton if it is either self-dual or anti-self-dual. On compact oriented 4-manifolds, an instanton is always an absolute minimizer of the Yang-Mills energy. Not all Yang-Mills connections are instantons. See [9, 10] for example the $SU(2)$ Yang-Mills connection on S^4 which are neither self-dual nor anti-self-dual.

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It is a very interesting problem for us to study when Yang-Mills connection is anti-self-dual (or self-dual).

In [1], Bourguignon-Lawson proved that Yang-Mills connection on $P \rightarrow X$ a principal $SU(N)$ -bundle, where X is a four-dimensional anti-self-dual compact Riemannian manifold with positive scalar curvature S must be anti-self-dual, provided $|F_A^+| \leq \frac{S}{12}$. It is extended to more general cases in [12, 13, 14].

Further, Min-Oo [7] proved that the Yang-Mills connection on $P \rightarrow X$ a principal $SU(2)$ -bundle must be anti-self-dual, whenever $\int_{S^4} |F_A^+|^2 \leq C$, where C is suitable positive constant.

Now we assume that the base manifold is a Kähler surface. Then the curvature splits into

$$F_A = F_A^{2,0} + F_A^{1,1} + F_A^{0,2},$$

where $F_A^{p,q}$ is the (p, q) -component. We have from the Bianchi identity $\partial_A F_A^{2,0} = 0$ and $\bar{\partial}_A F_A^{0,2} = 0$ with respect to the partial covariant derivatives. An anti-self-dual connection relates to a semi-stable holomorphic structure together with an Einstein-Hermitian structure on the associated complex vector bundle [2].

The self-dual part F_A^+ is given as $F_A^+ = F_A^{2,0} + F_A^{0,2} + \frac{1}{2}(\Lambda_\omega F_A) \otimes \omega$ and the anti-self-dual part F_A^- is a form of type $(1, 1)$ which is orthogonal to ω [3, 6].

Denote by \mathcal{A}_{YM} the space of Yang-Mills connections and \mathcal{A}_{HYM} the space of connections whose curvature satisfies $\hat{F}_A = \lambda Id_E$, here λ is a constant. These spaces are gauge invariant with respect to the group \mathcal{G} of gauge transformations.

The following gives an isolation phenomenon relative to L^2 -norm of \hat{F}_A .

Theorem 1.1. *Let M be a compact Kähler surface with positive scalar curvature and A be a Yang-Mills connection on a bundle E over M with compact, semi-simple Lie group G . There exist a positive constant $\delta = \delta(M, E, A)$ with the following significance. If \hat{F}_A satisfies*

$$\|\hat{F}_A - \lambda Id_E\|_{L^2(M)} \leq \delta,$$

where $\lambda = \frac{2(C_1(E) \cdot [w])}{\text{rank}(E)[w]^2}$, then

$$F_A^{0,2} = 0 \quad \text{and} \quad \hat{F}_A = \lambda Id_E$$

We have then from this theorem an open subset

$$W = \{[A] : \|\hat{F}_A - \lambda Id_E\|_{L^2(X)} \leq \delta\}$$

in the orbit space \mathcal{A}/\mathcal{G} of connections with property $\mathcal{A}_{HYM}/\mathcal{G} = W \cap \mathcal{A}_{YM}/\mathcal{G}$.

In the case that the scalar curvature $S = 0$, we consider irreducible Yang-Mills connections on compact Calabi-Yau 2-folds.

A connection is irreducible when it admits nontrivial covariantly constant Lie algebra-valued 0-form. Denote by $\hat{\mathcal{A}}_{YM}$ the space of irreducible Yang-Mills connections and $\hat{\mathcal{A}}_0$ the space of irreducible connections whose curvature satisfies $\hat{F}_A = 0$. Then the theorem asserts that $\hat{\mathcal{A}}_{YM} \cap \hat{\mathcal{A}}_0 = \hat{\mathcal{A}}_{ASD}$, here $\hat{\mathcal{A}}_{ASD}$ is the space of irreducible anti-self-dual connections. These spaces are gauge invariant with respect to the group \mathcal{G} of gauge transformations. So the moduli space of irreducible anti-self-dual connections $\hat{\mathcal{A}}_{ASD}/\mathcal{G} = \hat{\mathcal{A}}_{YM}/\mathcal{G} \cap \hat{\mathcal{A}}_0/\mathcal{G}$.

The following gives an isolation phenomenon relative to L^2 -norm of \hat{F}_A over compact Calabi-Yau 2-fold.

Theorem 1.2. *Let M be a compact simply-connected Calabi-Yau 2-fold and A be an irreducible Yang-Mills connection on a bundle E over M with compact, semi-simple Lie group G . There exist a positive constant $\delta = \delta(M, A)$ with the following significance. If \hat{F}_A satisfies*

$$\|\hat{F}_A\|_{L^2(M)} \leq \delta,$$

then A is anti-self-dual connection.

2 Yang-Mills connection over Kähler manifold

Let M be a Kähler manifold with Kähler metric g and $P \rightarrow M$ be a smooth principal bundle over M with a compact semi-simple Lie group G . For any connection A on P we have the covariant exterior derivatives $d_A : \Omega^k \rightarrow \Omega^{k+1}(\mathfrak{g}_P)$ where $\Omega^k(\mathfrak{g}_P)$ denotes the space of Lie-algebra value k -forms. Like the canonical splitting the exterior derivatives $d = \partial + \bar{\partial}$, decomposes over M into $d_A = \partial_A + \bar{\partial}_A$. The curvature splits into $F_A = F_A^{2,0} + F_A^{1,1} + F_A^{0,2}$, where $F_A^{p,q}$ is the (p, q) -component. Then we get from the Bianchi identity

$$\begin{aligned} \partial_A F_A^{2,0} &= \bar{\partial}_A F_A^{0,2} = 0 \\ \bar{\partial}_A F_A^{2,0} + \partial_A F_A^{1,1} &= \partial_A F_A^{0,2} + \bar{\partial}_A F_A^{1,1} = 0. \end{aligned}$$

Decompose the curvature, F_A , as

$$F_A = F_A^{2,0} + F_{A0}^{1,1} + \frac{1}{n} \hat{F}_A \otimes \omega + F_A^{0,2},$$

where $\hat{F}_A := \Lambda_\omega F_A$ and $F_{A0}^{1,1} = F_A^{1,1} - \frac{1}{n} \hat{F}_A \otimes \omega$.

Proposition 2.1. *Let A be a Yang-Mills connection on a bundle E over a Kähler n -fold, then*

$$(1) \quad 2\bar{\partial}_A^* F_A^{0,2} = \sqrt{-1} \bar{\partial}_A \hat{F}_A, \quad (2.1)$$

$$(2) \quad 2\partial_A^* F_A^{2,0} = -\sqrt{-1} \partial_A \hat{F}_A. \quad (2.2)$$

Proof. We only prove the first identity, the second's proof is similar. Recall that the Bianchi equations $d_A F_A = 0$, we take (1, 2) part, it implies that

$$0 = \bar{\partial}_A F_{A0}^{1,1} + \frac{1}{n} \bar{\partial}_A (\hat{F}_A \otimes \omega) + \partial_A F_A^{0,2},$$

hence,

$$0 = \sqrt{-1} \Lambda_\omega (\bar{\partial}_A F_{A0}^{1,1}) + \frac{1}{n} \sqrt{-1} \Lambda_\omega \bar{\partial}_A (\hat{F}_A \otimes \omega) + \sqrt{-1} \Lambda_w \partial_A F_A^{0,2}. \quad (2.3)$$

The Yang-Mills connection $d_A^* F_A = 0$, we take (0, 1) part, it implies that

$$\partial_A^* (F_{A0}^{1,1} + \frac{1}{n} \hat{F}_A \otimes \omega) + \bar{\partial}_A^* F_A^{0,2} = 0. \quad (2.4)$$

By Hodge identities (see [8] 6.12)

$$[\Lambda_\omega, \bar{\partial}_A] = -\sqrt{-1} \partial_A^* \quad \text{and} \quad [\Lambda_w, \partial_A] = \sqrt{-1} \bar{\partial}_A^*,$$

we can write (2.4) to

$$\begin{aligned} 0 &= \sqrt{-1} [\Lambda_\omega, \bar{\partial}_A] (F_{A0}^{1,1} + \frac{1}{n} \hat{F}_A \otimes \omega) - \sqrt{-1} [\Lambda_w, \partial_A] F_A^{0,2} \\ &= \sqrt{-1} \Lambda_\omega (\bar{\partial}_A F_{A0}^{1,1}) + \frac{1}{n} \sqrt{-1} \Lambda_\omega \bar{\partial}_A (\hat{F}_A \otimes \omega) - \sqrt{-1} \bar{\partial}_A \hat{F}_A - \sqrt{-1} \Lambda_w \partial_A F_A^{0,2}. \end{aligned} \quad (2.5)$$

By (2.3) and (2.5), we obtain

$$\begin{aligned} 0 &= -2\sqrt{-1} \Lambda_\omega \partial_A F_A^{0,2} - \sqrt{-1} \bar{\partial}_A \hat{F}_A \\ &= 2\bar{\partial}_A^* F_A^{0,2} - \sqrt{-1} \bar{\partial}_A \hat{F}_A. \end{aligned}$$

□

A connection is irreducible when it admits no nontrivial covariantly constant Lie algebra-value 0-form, i.e.,

$$\ker\{d_A : \Omega^0(\mathfrak{g}_P) \rightarrow \Omega^1(\mathfrak{g}_P)\} = \{0\}.$$

Lemma 2.2. *Let M be a compact Kähler n -fold ($n \geq 2$) and A be a irreducible Yang-Mills connection on a bundle E over M with compact, semi-simple Lie group G . There exist constants $C = C(\mathfrak{g}, A)$ such that either*

$$\hat{F}_A = 0,$$

or

$$\|\hat{F}_A\|_{L^2(M)} \leq C \|F_A^{0,2}\|_{L^4(M)}^2.$$

Proof. From Proposition 2.1 (2.1), we obtain

$$2\bar{\partial}_A\bar{\partial}_A^*F_A^{0,2} = \sqrt{-1}\bar{\partial}_A\bar{\partial}_A\hat{F}_A = \sqrt{-1}[F_A^{0,2}, \hat{F}_A].$$

Hence

$$\|\bar{\partial}_A^*F_A^{0,2}\|_{L^2(M)}^2 = \int_M \langle \bar{\partial}_A\bar{\partial}_A^*F_A^{0,2}, F_A^{0,2} \rangle = \int_M \langle \frac{\sqrt{-1}}{2}[F_A^{0,2}, \hat{F}_A], F_A^{0,2} \rangle \quad (2.6)$$

There is a constant $c > 0$ depending on the Lie algebra \mathfrak{g} such that $|[X, \bar{X}]| \leq c|X|^2$. Then

$$\begin{aligned} & \int_M \langle \frac{\sqrt{-1}}{2}[F_A^{0,2}, \hat{F}_A], F_A^{0,2} \rangle \\ &= - \int_M \text{tr}(\frac{\sqrt{-1}}{2}[F_A^{0,2}, \hat{F}_A] \wedge *F_A^{0,2}) = \int_M \text{tr}(\frac{\sqrt{-1}}{2}\hat{F}_A \wedge [F_A^{0,2}, *F_A^{0,2}]) \\ &= \int_M \langle \frac{\sqrt{-1}}{2}\hat{F}_A, \sum_{ij} [F_{ij}^{0,2}, \overline{F_{ij}^{0,2}}] \rangle \\ &\leq c \int_M |\hat{F}_A| |F_A^{0,2}|^2 \leq c \|\hat{F}_A\|_{L^2(M)} \|F_A^{0,2}\|_{L^4(M)}^2. \end{aligned} \quad (2.7)$$

here we used the Hölder inequality $\int_M |\hat{F}_A| |F_A^{0,2}|^2 \leq \|\hat{F}_A\|_{L^2(M)} \|F_A^{0,2}\|_{L^4(M)}^2$.

Then from (2.6) and (2.7), we get,

$$\|\bar{\partial}_A\hat{F}_A\|_{L^2(M)}^2 = 4\|\bar{\partial}_A^*F_A^{0,2}\|_{L^2(M)}^2 \leq c\|\hat{F}_A\|_{L^2(M)} \|F_A^{0,2}\|_{L^4(M)}^2 \quad (2.8)$$

In the similar way, we get

$$\|\partial_A\hat{F}_A\|_{L^2(M)}^2 \leq c\|\hat{F}_A\|_{L^2(M)} \|F_A^{2,0}\|_{L^4(M)}^2. \quad (2.9)$$

Since A is a irreducible connection, there exist a positive constant $\lambda = \lambda(A) \geq 0$ such that

$$\lambda\|\hat{F}_A\|_{L^2(M)} \leq \|d_A\hat{F}_A\|_{L^2(M)} = \|\nabla_A\hat{F}_A\|_{L^2(M)}.$$

From (2.8) and (2.9), we get

$$\begin{aligned} \lambda^2\|\hat{F}_A\|_{L^2(M)}^2 &\leq \|d_A\hat{F}_A\|_{L^2(M)}^2 = (\|\bar{\partial}_A\hat{F}_A\|_{L^2(M)}^2 + \|\partial_A\hat{F}_A\|_{L^2(M)}^2) \\ &\leq c\|\hat{F}_A\|_{L^2(M)} \|F_A^{0,2}\|_{L^4(M)}^2. \end{aligned}$$

Hence, we obtain

$$\|\hat{F}_A\|_{L^2(M)}^2 \leq c\lambda^{-2}\|\hat{F}_A\|_{L^2(M)} \|F_A^{0,2}\|_{L^4(M)}^2$$

Hence, we can choose $C = C(M, A) = c\lambda^{-2}$. □

Proposition 2.3. *Let M be a compact Kähler n -fold ($n \geq 2$) and A be a irreducible Yang-Mills connection on a bundle E over M with compact, semi-simple Lie group G . There exist a positive constant $\delta = \delta(M, A)$ with the following significance. If \hat{F}_A satisfies*

$$\|\hat{F}_A\|_{L^n(M)} \leq \delta,$$

then

$$\ker \Delta_{\bar{\partial}_A}|_{\Omega^0(\mathfrak{g}_P)} = \{0\}.$$

Proof. By [3] Lemma 6.1.7, for any connection A , we have

$$2\bar{\partial}_A^* \bar{\partial}_A s = d_A^* d_A s + \sqrt{-1}[\hat{F}_A, s],$$

where $s \in \Omega^0(\mathfrak{g}_P)$. For any section $s \in \ker \Delta_{\bar{\partial}_A}$, we have

$$d_A^* d_A s = -\sqrt{-1}[\hat{F}_A, s],$$

hence,

$$\begin{aligned} \|d_A s\|_{L^2(M)}^2 &= - \int_M \langle \sqrt{-1}[\hat{F}_A, s], s \rangle \\ &\leq \|\hat{F}_A\|_{L^n(M)} \|s\|_{L^{\frac{2n}{n-1}}(M)}^2 \leq C_S \|\hat{F}_A\|_{L^n(M)} \|s\|_{L_1^2(M)}^2 \end{aligned} \quad (2.10)$$

The last inequality, we used the Sobolev embedding $L_1^2(M) \hookrightarrow L^{\frac{2n}{n-1}}(M)$ with embedding constant C_S , here $\dim(M) = 2n$.

Since A is a irreducible connection, then there exist a positive constant $\lambda = \lambda(A)$ such that

$$\lambda \|s\|_{L^2(M)} \leq \|d_A s\|_{L^2(M)} = \|\nabla_A s\|_{L^2(M)}. \quad (2.11)$$

By Kato inequality, $|\nabla|s|| \leq |\nabla_A s|$ and (2.10), (2.11), hence

$$\|d_A s\|_{L^2(M)}^2 \leq C_S \|\hat{F}_A\|_{L^n(M)} (1 + \lambda^{-2}) \|d_A s\|_{L^2(M)}^2$$

We can choose $\delta = \frac{1}{2C_S(1+\lambda^{-2})}$, then $d_A s \equiv 0$. Since A is a irreducible connection, we obtain $s \equiv 0$. \square

3 Yang-Mills connection over a Kähler surface

We set $\Delta_{\bar{\partial}_A} = \bar{\partial}_A \bar{\partial}_A^* + \bar{\partial}_A^* \bar{\partial}_A$, $\Delta_{\partial_A} = \partial_A \partial_A^* + \partial_A^* \partial_A$, and $\Delta_A = d_A^* d_A + d_A d_A^*$.

Proposition 3.1. (Weitzenböck formula) *Let M be a complete Kähler surface with Riemannian metric g and A be a connection on a bundle E over M . Then for each $\phi \in \Omega^{0,2}(\mathfrak{g}_P^{\mathbb{C}})$,*

$$\Delta_{\bar{\partial}_A} \phi = \nabla_A^* \nabla_A \phi + \sqrt{-1}[\hat{F}_A, \phi] + 2S\phi \quad (3.1)$$

where S is the scalar curvature of the metric g .

Proof. The component of $\bar{\partial}_A \phi$ is given by $\nabla_{A,\bar{\mu}} \phi_{\bar{\nu}\bar{\lambda}} + \nabla_{A,\bar{\nu}} \phi_{\bar{\lambda}\bar{\mu}} + \nabla_{A,\bar{\lambda}} \phi_{\bar{\mu}\bar{\nu}}$, where $\nabla_{A,\bar{\mu}} \cdot = \nabla_{\bar{\mu}} \cdot + [A_{\bar{\mu}}, \cdot]$ ($\nabla_{A,\mu} \cdot = \nabla_{\mu} \cdot + [A_{\mu}, \cdot]$). Then $\bar{\partial}_A^* \bar{\partial}_A \phi$ reduces to

$$- \sum g^{\sigma\bar{\tau}} \nabla_{A,\sigma} (\nabla_{A,\bar{\tau}} \phi_{\bar{\mu}\bar{\nu}} + \nabla_{A,\bar{\mu}} \phi_{\bar{\nu}\bar{\tau}} + \nabla_{A,\bar{\nu}} \phi_{\bar{\tau}\bar{\mu}}).$$

We have similarly

$$\bar{\partial}_A \bar{\partial}_A^* \phi = - \sum g^{\sigma\bar{\tau}} (\nabla_{A,\bar{\mu}} \nabla_{A,\sigma} \phi_{\bar{\tau}\bar{\nu}} - \nabla_{A,\bar{\nu}} \nabla_{A,\sigma} \phi_{\bar{\tau}\bar{\mu}}).$$

Then

$$\begin{aligned} (\Delta_{\bar{\partial}_A} \phi)_{\bar{\mu}\bar{\nu}} &= - \sum g^{\sigma\bar{\tau}} \nabla_{A,\sigma} \nabla_{A,\bar{\tau}} \phi_{\bar{\mu}\bar{\nu}} - \sum g^{\sigma\bar{\tau}} [\nabla_{A,\bar{\mu}}, \nabla_{A,\sigma}] \phi_{\bar{\tau}\bar{\nu}} \\ &\quad + \sum g^{\sigma\bar{\tau}} [\nabla_{A,\bar{\nu}}, \nabla_{A,\sigma}] \phi_{\bar{\tau}\bar{\mu}} \\ &= - \sum g^{\sigma\bar{\tau}} \nabla_{A,\sigma} \nabla_{A,\bar{\tau}} \phi_{\bar{\mu}\bar{\nu}} - \sum g^{\sigma\bar{\tau}} [F_{A,\bar{\mu}\sigma}, \phi_{\bar{\tau}\bar{\nu}}] \\ &\quad + \sum g^{\sigma\bar{\tau}} [F_{A,\bar{\nu}\sigma}, \phi_{\bar{\tau}\bar{\mu}}] + \sum (R_{\bar{\mu}}^{\bar{\gamma}} \phi_{\bar{\gamma}\bar{\nu}} - R_{\bar{\nu}}^{\bar{\gamma}} \phi_{\bar{\gamma}\bar{\mu}}). \end{aligned}$$

Since the base manifold is Kähler surface $S = \frac{1}{2} \sum g^{\bar{\sigma}\tau} R_{\tau\bar{\sigma}}$ and $\hat{F}_A = \sqrt{-1} \sum g^{\tau\bar{\sigma}} F_{A,\bar{\sigma}\tau}$. Thus (3.1) is obtained. \square

From Proposition 2.1 equation (2.1), we have

$$\Delta_{\bar{\partial}_A} F_A^{0,2} = -\frac{1}{2} [\sqrt{-1} \hat{F}_A, F_A^{0,2}],$$

then we obtain

Proposition 3.2. *Let M be a complete Kähler surface with Riemannian metric g and A be a Yang-Mills connection on a bundle E over M . Then we have*

$$\nabla_A^* \nabla_A F_A^{0,2} + \frac{3}{2} [\sqrt{-1} \hat{F}_A, F_A^{0,2}] + 2S F_A^{0,2} = 0 \quad (3.2)$$

From the identity,

$$*F_A = F_A^+ - F_A^- = F_A^{0,2} + \frac{1}{2} \hat{F}_A \otimes \omega - F_{A_0}^{1,1} + F_A^{0,2}.$$

We can write Yang-Mills functional as

$$\begin{aligned} YM(A) &= 4\|F_A^{0,2}\|^2 + \|\hat{F}_A\|^2 + (2C_2(E) - C_1(E)^2) \\ &= 4\|F_A^{0,2}\|^2 + \|\hat{F}_A - \lambda Id_E\|^2 + (2C_2(E) - C_1(E)^2) + \frac{2(C_1(E) \cdot [\omega])^2}{rank(E)[\omega]^2}, \end{aligned}$$

where $\lambda = \frac{2(C_1(E) \cdot [\omega])}{rank(E)[\omega]^2}$. The energy functional $\|\hat{F}_A\|^2$ plays an important role in the study of Hermitian-Einstein connections (See [2] and [11]). Recall that a connection on a holomorphic vector bundle on a Kähler manifold is called Hermitian-Einstein if $\hat{F}_A = \lambda Id$.

Proposition 3.3. ([2] Proposition 3) *Let A be an integrable Yang-Mills connection on an Hermitian vector bundle E over a Kähler n -fold M . Then A is a direct sum of Hermitian-Einstein connections. Further more, we can denote $A = \oplus_{i=1}^l A_i$ where $E = \oplus_{i=1}^l E_i$ is an orthogonal splitting of E , and where $\hat{F}_{A_i} = \lambda_i Id_{E_i}$.*

Under condition of Proposition 3.3, we have

$$\|\hat{F}_A - \lambda Id_E\|_{L^2(M)} = \left(\sum_{i=1}^l \text{rank}(E_i) |\lambda_i - \lambda|^2 \text{Vol}(M) \right)^{\frac{1}{2}}.$$

Corollary 3.4. *Let A be an integrable Yang-Mills connection on an Hermitian vector bundle E over a Kähler n -fold M . Then there exist a constant $\delta = \delta(M, E, A)$ such that*

$$\|\hat{F}_A - \lambda Id_E\|_{L^2(M)} \geq \delta,$$

or

$$\hat{F}_A = \lambda Id_E$$

Proof. If we suppose the Yang-Mills connection A is not a Hermitian-Yang-Mills connection. Then from Proposition 3.3, the orthogonal splitting, $\oplus_{i=1}^l E_i$, of E such that there exist $\lambda_i \neq \lambda$. Hence

$$\|\hat{F}_A - \lambda Id_E\|_{L^2(M)} \geq \left(\text{rank}(E_i) |\lambda_i - \lambda|^2 \text{vol}(M) \right)^{\frac{1}{2}}.$$

then we can choose $\delta = (\text{rank}(E_i) |\lambda_i - \lambda|^2 \text{vol}(M))^{\frac{1}{2}}$. \square

Next, we gives an isolation theorem relative to L^2 -norm of \hat{F}_A on a compact Kähler surface with positive scalar curvature as follow.

Theorem 3.5. *Let M be a compact Kähler surface with positive scalar curvature and A be a Yang-Mills connection on a bundle E over M with compact, semi-simple Lie group G . There exist a positive constant $\delta = \delta(M, E, A)$ with the following significance. If \hat{F}_A satsfies*

$$\|\hat{F}_A - \lambda Id_E\|_{L^2(M)} \leq \delta,$$

where $\lambda = \frac{2(C_1(E) \cdot [w])}{\text{rank}(E)[w]^2}$, then

$$F_A^{0,2} = 0 \quad \text{and} \quad \hat{F}_A = \lambda Id_E$$

Proof. From Proposition 3.2, we have

$$\nabla_A^* \nabla_A F_A^{0,2} + \frac{3\sqrt{-1}}{2} [\hat{F}_A - \lambda Id, F_A^{0,2}] + 2S F_A^{0,2} = 0$$

and hence

$$\int_M \langle \nabla_A^* \nabla_A F_A^{0,2}, F_A^{0,2} \rangle + 2S \int_M \langle F_A^{0,2}, F_A^{0,2} \rangle = - \int_M \langle \frac{3\sqrt{-1}}{2} [\hat{F}_A - \lambda Id, F_A^{0,2}], F_A^{0,2} \rangle \quad (3.3)$$

By Kato inequality, $|\nabla|F_A^{0,2}|| \leq |\nabla_A F_A^{0,2}|$, we estimate left hand of (3.3)

$$\int_M \langle \nabla_A^* \nabla_A F_A^{0,2}, F_A^{0,2} \rangle + 2S \int_M \langle F_A^{0,2}, F_A^{0,2} \rangle \geq C \|F_A^{0,2}\|_{L_1^2(X)}^2, \quad (3.4)$$

where $C = \min\{1, 2S\}$. Next, we estimate right hand of (3.3)

$$| \int_M \langle \frac{3\sqrt{-1}}{2} [\hat{F}_A - \lambda Id, F_A^{0,2}], F_A^{0,2} \rangle | \leq \|\hat{F}_A - \lambda Id_E\|_{L^2(X)} \|F_A^{0,2}\|_{L^4(X)} \quad (3.5)$$

Then from (3.3)–(3.5), we get

$$\begin{aligned} C \|F_A^{0,2}\|_{L_1^2(X)}^2 &\leq \|\hat{F}_A - \lambda Id_E\|_{L^2(X)} \|F_A^{0,2}\|_{L^4(X)} \\ &\leq C_S \delta \|F_A^{0,2}\|_{L_1^2(X)}. \end{aligned}$$

where C_S is Sobolev constant. We choose $\delta = \frac{C}{2C_S}$, then $F_A^{0,2} \equiv 0$. From Corollary 3.4, we also obtain $\hat{F}_A = \lambda Id_E$. \square

From Proposition 3.2, for each $\phi \in \ker \Delta_{\bar{\partial}_A}|_{\Omega^{0,2}(\mathfrak{g}_P^{\mathbb{C}})}$, we have

$$\nabla_A^* \nabla_A \phi + \sqrt{-1} [\hat{F}_A - \lambda Id_E, \phi] + 2S\phi = 0.$$

As above, we have

Corollary 3.6. *Let M be a compact Kähler surface with positive scalar curvature and A be a Yang-Mills connection on a bundle E over M with compact, semi-simple Lie group G . There exist a positive constant $\delta = \delta(M)$ with the following significance. If \hat{F}_A satisfies*

$$\|\hat{F}_A - \lambda Id_E\|_{L^2(M)} \leq \delta,$$

where $\lambda = \frac{2(C_1(E) \cdot [w])}{\text{rank}(E)[w]^2}$, then

$$\ker \Delta_{\bar{\partial}_A}|_{\Omega^{0,2}(\mathfrak{g}_P^{\mathbb{C}})} = \{0\}.$$

In the case that the scalar curvature $S = 0$, we consider irreducible Yang-Mills connections on compact Calabi-Yau 2-folds.

Let M be a compact simply-connected Calabi-Yau 2-fold, with Kähler form ω and nonzero covariant constant $(2, 0)$ -form θ ([4] Definition 4.3 and [5] Corollary 4.B.23), here $\theta \wedge \bar{\theta} = \frac{\omega^2}{2}$. The form θ give us a Hodge star

$$*_\theta : \Omega^{0,2}(\mathfrak{g}_P) \rightarrow \Omega^0(\mathfrak{g}_P)$$

defined by $*_{\theta} \cdot = *(\cdot \wedge \theta)$

Let A be a connection on a G -bundle E over M . We can define a section $s \in \Omega^0(\mathfrak{g}_P^{\mathbb{C}})$, such that

$$*(s \wedge \theta) = F_A^{0,2}, \quad (3.6)$$

hence, we have

$$-*(\bar{\partial}_A s \wedge \theta) = \bar{\partial}_A^* F_A^{0,2}. \quad (3.7)$$

From (3.6) and (3.7), in a direct calculate, we have

$$|s| = |F_A^{0,2}| \quad \text{and} \quad |\bar{\partial}_A s| = |\bar{\partial}_A^* F_A^{0,2}|.$$

Next, we also give an isolation theorem relative to L^2 -norm of \hat{F}_A on compact Calabi-Yau 2-folds.

Theorem 3.7. *Let M be a compact simply-connected Calabi-Yau 2-fold and A be a irreducible Yang-Mills connection on a bundle E over M with compact, semi-simple Lie group G . There exist a positive constant $\delta = \delta(M, A)$ with the following significance. If \hat{F}_A satisfies*

$$\|\hat{F}_A\|_{L^2(M)} \leq \delta,$$

then A is anti-self-dual.

Proof. By [3] Lemma 6.1.7, for any connection A , we have

$$2\bar{\partial}_A^* \bar{\partial}_A s = d_A^* d_A s + \sqrt{-1}[\hat{F}_A, s],$$

where $s \in \Omega^0(\mathfrak{g}_P^{\mathbb{C}})$ satisfies (3.6). Hence, we have

$$\|d_A s\|_{L^2(M)}^2 = 2\|\bar{\partial}_A s\|_{L^2(M)}^2 - \int_M \langle \sqrt{-1}[\hat{F}_A, s], s \rangle. \quad (3.8)$$

We estimate the right hand of (3.8),

$$\begin{aligned} & 2\|\bar{\partial}_A s\|_{L^2(M)}^2 - \int_M \langle \sqrt{-1}[\hat{F}_A, s], s \rangle \\ &= 2\|\bar{\partial}_A^* F_A^{0,2}\|_{L^2(M)}^2 - \int_M \langle \sqrt{-1}[\hat{F}_A, s], s \rangle \\ &= 2 \int_M \langle \bar{\partial}_A \bar{\partial}_A^* F_A^{0,2}, F_A^{0,2} \rangle - \int_M \langle \sqrt{-1}[\hat{F}_A, s], s \rangle \\ &= \int_M \langle \sqrt{-1} \bar{\partial}_A \bar{\partial}_A^* \hat{F}_A, F_A^{0,2} \rangle - \int_M \langle \sqrt{-1}[\hat{F}_A, s], s \rangle \\ &\leq 2\|\hat{F}_A\|_{L^2(M)} \|F_A^{0,2}\|_{L^4(M)}^2 + 2\|\hat{F}_A\|_{L^2(M)}^2 \|s\|_{L^4(M)}^2 \end{aligned} \quad (3.9)$$

By Kato inequality, $|\nabla|s|| \leq |\nabla_A s| = |d_A s|$, we have

$$\|F_A^{0,2}\|_{L^4(M)}^2 = \|s\|_{L^4(M)}^2 \leq C_S \|s\|_{L_1^2(M)}^2 \leq C_S (\|s\|_{L^2(M)}^2 + \|d_A s\|_{L^2(M)}^2). \quad (3.10)$$

Since A is a irreducible connection, then there exist a positive constant $\lambda = \lambda(A)$ such that

$$\lambda \|s\|_{L^2(M)} \leq \|d_A s\|_{L^2(M)}. \quad (3.11)$$

From (3.9)–(3.11), we obtain

$$\|d_A s\|_{L^2(M)}^2 \leq 4C_S \|\hat{F}_A\|_{L^2(M)} (1 + \lambda^{-2}) \|d_A s\|_{L^2(M)}^2.$$

We choose $\delta = \frac{1}{8C_S(1+\lambda^{-2})}$, then $d_A s \equiv 0$. Since A is irreducible, then $s \equiv 0$. \square

From Proposition 2.2, we have

Corollary 3.8. *Let M be a compact simply-connected Calabi-Yau 2-fold and A be a irreducible Yang-Mills connection on a bundle E over M with compact, semi-simple Lie group G . There exist a positive constant $\delta = \delta(M, A)$ with the following significance. If $F_A^{0,2}$ satisfies*

$$\|F_A^{0,2}\|_{L^4(M)} \leq \delta,$$

then A is anti-self-dual connection.

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